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Theta vectors and quantum theta functions

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Abstract

In this paper, we clarify the relation between Manin's quantum theta function and Schwarz's theta vector. We do this in comparison with the relation between the kq representation, which is equivalent to the classical theta function, and the corresponding coordinate space wavefunction. We first explain the equivalence relation between the classical theta function and the kq representation in which the translation operators of the phase space are commuting. When the translation operators of the phase space are not commuting, then the kq representation is no longer meaningful. We explain why Manin's quantum theta function, obtained via algebra (quantum torus) valued inner product of the theta vector, is a natural choice for the quantum version of the classical theta function. We then show that this approach holds for a more general theta vector containing an extra linear term in the exponent obtained from a holomorphic connection of constant curvature than the simple Gaussian one used in Manin's construction.

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1. Introduction

Classical theta functions can be regarded as state functions on classical tori, and have played an important role in the string loop calculation [1, 2]. Its quantum version on the noncommutative tori has been discussed mainly by Manin [3–5] and Schwarz [6, 7]. In the physics literature, it has been discussed in the context of noncommutative soliton [8].

In noncommutative field theory, one can find nontrivial soliton solutions in terms of projection operators [8–10]. Before this development, Boca [11] had constructed projection operators on the $\mathbb{Z}_4$-orbifold of noncommutative two torus. There it was also shown that these projection operators can be expressed in terms of the classical theta functions, of which certain classical commuting variables are replaced with quantum operators. Elicited from and generalizing Boca's result, Manin [4, 5] explicitly constructed a quantum theta function, the concept of which he introduced previously [3]. In both Boca's and Manin's constuctions, the main pillars were the algebra valued inner product that Rieffel [12] used in his classic work on projective modules over noncommutative tori. One major difference is that in Manin's construction of the quantum theta function, the so-called theta vector that Schwarz introduced earlier [6, 7] was used for the inner product, while in Boca's construction the eigenfunctions of Fourier transform were used.

Both the classical theta function [13] and the kq representation in the physics literature [14, 15] have been known for a long time. The kq representation is a transformation of a wavefunction on (real n-dimensional) coordinate space to a function on (real $2n$-dimensional) phase space consisting of (quasi-)coordinates and (quasi-)momenta. However, the translation operators in the kq representation acting on the lattice of the phase space are commuting. When the lattice of the phase space is periodic, one can identify functions possessing translational symmetry on the lattice with the classical theta functions on tori. When the translation operators of the coordinate and momentum directions are not commuting, the kq representation and the classical theta function lose their meaning. One has to find other ways of representing periodic functions on the lattice of the noncommuting phase space. When the algebras are noncommutative, the algebra valued inner product is a good fit for constructing operators out of state functions. In the case at hand, the coordinates of the phase space are noncommuting and so is the algebra based on them. And the functions on the noncommuting phase space can be regarded as operators.

In fact, the theta vector corresponds to a state on a quantum torus, and the quantum theta function defined by Manin [4, 5] is an operator acting on the states (module) on a quantum torus. Therefore, it is very natural to use the algebra valued inner product to build the quantum theta functions from the theta vectors over noncommutative tori. The classical theta function possesses a certain symmetry property under the lattice translation, and Manin's quantum theta function is constructed in such a way that this symmetry property is maintained as a functional relation, which it should satisfy.

The organization of the paper is as follows. In section 2, we review the classical theta function briefly, then explain the relationship between the classical theta function and the kq representation. In section 3, we first review the theta vectors on quantum tori, then explain how the concept of Manin's quantum theta function emerges from algebra valued inner product of a state function. In section 4, we provide further support for Manin's approach by applying it to the case of a more general theta vector containing a constant satisfying the holomorphy conditon, and show that the new quantum theta function also satisfies Manin's functional relation for the consistency requirement. In section 5, we conclude with a discussion.

2. Classical complex tori and kq representation

In this section, we discuss the relationship between the classical theta function and the so-called kq representation [14, 15]. We first look into how the classical theta function emerges from Gaussian function via Fourier-like transformation. We then show that the transformed function is exactly equivalent to the kq representation known in the physics literature.

We now recall the property of the classical theta function briefly, then show how Gaussian function can be transformed into the classical theta function. The classical theta function Θ is a complex valued function on $\mathbb{C}^n$ satisfying the following relation:

$$\Theta(z+\lambda') = \Theta(z) \qquad \text{for} \quad z \in \mathbb{C}^n, \quad \lambda' \in \Lambda', \tag{1}$$

$$\Theta(z+\lambda) = c(\lambda)\,\mathrm{e}^{q(\lambda,z)}\Theta(z) \qquad \text{for} \quad \lambda \in \Lambda, \tag{2}$$

where $\Lambda' \bigoplus \Lambda \subset \mathbb{C}^n$ is a discrete sublattice of rank $2n$ split into the sum of two sublattices of rank n, isomorphic to $\mathbb{Z}^n$, and $c : \Lambda \to \mathbb{C}$ is a map and $q : \Lambda \times \mathbb{C} \to \mathbb{C}$ is a biadditive pairing linear in z.

The function $\Theta(z, T)$ satisfying (1) and (2) is defined as

$$\Theta(z, T) = \sum_{k \in \mathbb{Z}^n} \mathrm{e}^{\pi \mathrm{i}(k^t T k + 2k^t z)} \tag{3}$$

where T is a symmetric complex valued $n \times n$ matrix whose imaginary part is positive definite. Let $f_T(x)$ be a Gaussian function defined below using the same T above.

$$f_T(x) = \mathrm{e}^{\pi \mathrm{i} x^t T x} \qquad \text{for} \quad x \in \mathbb{R}^n. \tag{4}$$

Then $\widetilde{f}_T(\rho, \sigma)$ is defined as [6]

$$\widetilde{f}_T(\rho, \sigma) \equiv \sum_{k \in \mathbb{Z}^n} \mathrm{e}^{-2\pi \mathrm{i} \rho^t k} f_T(\sigma + k) \tag{5}$$

where $\rho, \sigma \in \mathbb{R}^n$. When we fix σ, this is a Fourier transformation between k and ρ. Then from (5), we get $\Theta(z, T)$ with a substitution $z = T\sigma - \rho$ as follows:

$$\widetilde{f}_T(\rho, \sigma) = \sum_{k \in \mathbb{Z}^n} \mathrm{e}^{\pi \mathrm{i}((\sigma+k)^t T(\sigma+k) - 2\rho^t k)} \tag{6}$$

$$= \mathrm{e}^{\pi \mathrm{i} \sigma^t T \sigma} \Theta(T\sigma - \rho, T). \tag{7}$$

We can do the same procedure for a general Gaussian function, $f_{T,c}(x)$, as follows:

$$f_{T,c}(x) = \mathrm{e}^{\pi \mathrm{i}(x^t T x + 2c^t x)} \tag{8}$$

where $c \in \mathbb{C}^n$. Then,

$$\widetilde{f}_{T,c}(\rho, \sigma) \equiv \sum_{k \in \mathbb{Z}^n} \mathrm{e}^{-2\pi \mathrm{i} \rho^t k} f_{T,c}(\sigma + k) \tag{9}$$

$$= \mathrm{e}^{\pi \mathrm{i}(\sigma^t T \sigma + 2c^t \sigma)} \Theta(T\sigma - \rho + c, T). \tag{10}$$

In this case we get $\Theta(z, T)$ with a substitution $z = T\sigma - \rho + c$.

The transformation (5) exactly matches the transformation used in defining the kq representation which has already appeared in the physics literature [14, 15].

The kq representation which defines symmetric coordinates k (quasimomentum) and q (quasicoordinate) is a transformation from a wavefunction in position space into a wavefunction in both k and q, which we denote as $C(k, q)$. $C(k, q)$ is defined by [15]

$$C(k, q) = \left(\frac{a}{2\pi}\right)^{\frac{1}{2}} \sum_{l \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}kal} \psi(q - la) \tag{11}$$

where a is a real number (lattice constant), and the 'coordinates' of the phase space (k, q) run over the intervals $-\frac{\pi}{a} < k \leqslant \frac{\pi}{a}$ and $-\frac{a}{2} < q \leqslant \frac{a}{2}$. In this representation, the displacement operators $\mathrm{e}^{\mathrm{i}mbx}, \mathrm{e}^{\mathrm{i}nap}$ in the x and p directions, where $[x, p] = i$, $b = \frac{2\pi}{a}$ and $m, n \in \mathbb{Z}$, are mutually commuting, and thus they simply become simple multiplication by the function $\mathrm{e}^{\mathrm{i}m\frac{2\pi}{a}q}$ and $\mathrm{e}^{\mathrm{i}nak}$, respectively [15].

Comparing (11) with (5), it is not difficult to see that $C(k, q)$ corresponds to $\widetilde{f}_T(\rho, \sigma)$ in our previous discussion with a correspondence $(\rho \leftrightarrow k)$ and $(\sigma \leftrightarrow q)$. Furthermore, from (11) it can be easily checked that

$$C\left(k + \frac{2\pi}{a}, q\right) = C(k, q), \tag{12}$$

$$C(k, q + a) = e^{ika}C(k, q). \tag{13}$$

These exactly match (1) and (2), the property of the classical theta function. We can thus say that the classical theta function corresponds to the kq representation, $C(k, q)$, while the pre-transformed Gaussian function $f_T(x)$ for the classical theta function corresponds to the wavefunction $\psi(x)$ for the kq representation. This correspondence is only valid when the lattice translation operators of the phase space (x, p) are mutually commuting.

3. Theta vectors on quantum torus and algebra valued inner product for a passage to quantum theta functions

In this section, we first discuss theta vectors on quantum torus and define the algebra (quantum torus) valued inner product on the modules over the quantum torus. Then we introduce the concept of Manin's quantum theta function [5] via the algebra valued inner product.

A noncommutative d-torus T_θ^d is a C^*-algebra generated by d unitaries $U_1, \ldots, U_d$ subject to the relations

$$U_\alpha U_\beta = e^{2\pi i\theta_{\alpha\beta}}U_\beta U_\alpha, \qquad \text{for} \quad 1 \leqslant \alpha, \beta \leqslant d, \tag{14}$$

where $\theta = (\theta_{\alpha\beta})$ is a skew symmetric matrix with real entries.

Let L be all derivations on T_θ^d, i.e.,

$$L = \{\delta|\delta : T_\theta^d \to T_\theta^d, \text{which is linear, and } \delta(fg) = \delta(f)g + f\delta(g)\}.$$

Then L has a Lie algebra structure since $[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1 \in L$. We can also see that L is isomorphic to $\mathbb{R}^d$. A noncommutative torus is said to have a complex structure if the Lie algebra $L = \mathbb{R}^d$ acting on T_θ^d is equipped with the complex structure that we explain below. A complex structure on L can be considered a decomposition of complexification $L \bigoplus iL$ of L into a direct sum of two complex conjugate subspace $L^{1,0}$ and $L^{0,1}$. We denote a basis in L by $\delta_1, \ldots, \delta_d$, and a basis in $L^{0,1}$ by $\tilde{\delta}_1, \ldots, \tilde{\delta}_n$ where $d = 2n$. One can express $\tilde{\delta}_\alpha$ in terms of δ_j as $\tilde{\delta}_\alpha = t_{\alpha j}\delta_j$, where $t_{\alpha j}$ is a complex $n \times d$ matrix.

Let ∇_j (for $j = 1, \ldots, d$) be a constant curvature connection on a T_θ^d-module $\mathcal{E}$. A complex structure on $\mathcal{E}$ can be defined as a collection of $\mathbb{C}$ linear operators $\tilde{\nabla}_1, \ldots, \tilde{\nabla}_n$ satisfying

$$\tilde{\nabla}_\alpha (a \cdot f) = a\tilde{\nabla}_\alpha f + (\tilde{\delta}_\alpha a) \cdot f \tag{15}$$

$$[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] = 0 \tag{16}$$

where $a \in T_\theta^d$ and $f \in \mathcal{E}$ [6].

These two conditions are satisfied if we choose $\tilde{\nabla}_\alpha$ as

$$\tilde{\nabla}_\alpha = t_{\alpha j}\nabla_j \qquad \text{for} \quad \alpha = 1, \ldots, n, \quad j = 1, \ldots, d. \tag{17}$$

A vector $f \in \mathcal{E}$ is holomorphic if

$$\tilde{\nabla}_\alpha f = 0, \qquad \text{for} \quad \alpha = 1, \ldots, n. \tag{18}$$

A finitely generated projective module over T_θ^d can take the form $S(\mathbb{R}^p \times \mathbb{Z}^q \times F)$ where $2p + q = d$ and F is a finite Abelian group [12]. Here, $S(M)$ denotes the Schwartz functions on M which rapidly decay at infinity.

Here, we consider the case that the module is given by $S(\mathbb{R}^n)$, and choose a constant curvature connection ∇ on $S(\mathbb{R}^n)$ such that

$$(\nabla_\alpha, \nabla_{n+\alpha}) = \left(\frac{\partial}{\partial x^\alpha}, -2\pi i\sigma_\alpha x_\alpha\right) \qquad \text{for} \quad \alpha = 1, \ldots, n, \tag{19}$$

where σ_α are some real constants, x^α are coordinate functions on $\mathbb{R}^n$ and repeated indices are not summed. Then the curvature $[\nabla_i, \nabla_j] = F_{ij}$ satisfies $F_{\alpha,n+\alpha} = 2\pi\mathrm{i}\sigma_\alpha$, $F_{n+\alpha,\alpha} = -2\pi\mathrm{i}\sigma_\alpha$ and all others are zero. Now, we change the coordinates such that $t = (t_{\alpha j})$ becomes

$$t = (\mathbf{1}, \tau), \tag{20}$$

where $\mathbf{1}$ is an identity matrix of size n and τ is an $n \times n$ complex valued matrix.

Then, the holomorphic vector f satisfying (18) can be expressed as

$$\left(\frac{\partial}{\partial x^\alpha} - \sum_\beta 2\pi\mathrm{i}T_{\alpha\beta}x^\beta\right) f = 0, \tag{21}$$

where the $n \times n$ matrix $T = (T_{\alpha\beta})$ is given as follows. Condition (16) requires that the matrix T be symmetric, $T_{\alpha\beta} = T_{\beta\alpha}$, and it is given by $T_{\alpha\beta} = \tau_{\alpha\beta}\sigma_\beta$, $\alpha, \beta = 1, \ldots, n$, with the repeated index β not summed. Up to a constant we get,

$$f(x^1, \ldots, x^n) = \mathrm{e}^{\pi\mathrm{i}x^\alpha T_{\alpha\beta}x^\beta}. \tag{22}$$

If $\mathrm{Im}\, T$ is positive definite, then f belongs to $S(\mathbb{R}^n)$. The vectors satisfying the holomorphy condition (18) are called the theta vectors [6].

If a constant in $\mathbb{C}^n$ is added to a given connection $\widetilde{\nabla}$, it still yields the same constant curvature. Then the holomorphy condition (18) becomes [7, 16]

$$(\widetilde{\nabla}_\alpha - 2\pi\mathrm{i}c_\alpha)f_c = 0, \qquad \text{for} \quad \alpha = 1, \ldots, n \tag{23}$$

for $f_c \in S(\mathbb{R}^n)$, giving the following condition:

$$\left(\frac{\partial}{\partial x^\alpha} - \sum_\beta 2\pi\mathrm{i}T_{\alpha\beta}x^\beta - 2\pi\mathrm{i}c_\alpha\right) f_c = 0, \tag{24}$$

whose solution we get

$$f_c(x) = \mathrm{e}^{\pi\mathrm{i}x^\alpha T_{\alpha\beta}x^\beta + 2\pi\mathrm{i}c_\alpha x^\alpha}. \tag{25}$$

Now, we consider the algebra valued inner product on a bimodule after Rieffel [12]. Let M be any locally compact Abelian group, $\widehat{M}$ be its dual group and $\mathcal{G} \equiv M \times \widehat{M}$. Let π be a representation of $\mathcal{G}$ on $L^2(M)$ such that

$$\pi_x\pi_y = \alpha(x, y)\pi_{x+y} = \alpha(x, y)\overline{\alpha}(y, x)\pi_y\pi_x \qquad \text{for} \quad x, y \in \mathcal{G} \tag{26}$$

where α is a map $\alpha : \mathcal{G} \times \mathcal{G} \to \mathbb{C}^*$ satisfying

$$\alpha(x, y) = \alpha(y, x)^{-1}, \qquad \alpha(x_1 + x_2, y) = \alpha(x_1, y)\alpha(x_2, y),$$

and $\overline{\alpha}$ denotes the complex conjugation of α.

Let D be a discrete subgroup of $\mathcal{G}$. We define $S(D)$ as the space of Schwartz functions on D. For $\Phi \in S(D)$, it can be expressed as $\Phi = \sum_{w\in D} \Phi(w)e_{D,\alpha}(w)$ where $e_{D,\alpha}(w)$ is a delta function with support at w and obeys the following relation:

$$e_{D,\alpha}(w_1)e_{D,\alpha}(w_2) = \alpha(w_1, w_2)e_{D,\alpha}(w_1 + w_2). \tag{27}$$

For Schwartz functions $f, g \in S(M)$, the algebra $(S(D))$ valued inner product is defined as

$${}_D\langle f, g\rangle \equiv \sum_{w\in D} {}_D\langle f, g\rangle(w)e_{D,\alpha}(w) \tag{28}$$

where

$${}_D\langle f, g\rangle(w) = \langle f, \pi_w g\rangle.$$

Here, the scalar product of the type $\langle f, p\rangle$ used above for $f, p \in L^2(M)$ denotes the following:

$$\langle f, p\rangle = \int f(x_1)\overline{p(x_1)}\,\mathrm{d}\mu_{x_1} \qquad \text{for} \quad x = (x_1, x_2) \in M \times \widehat{M}, \tag{29}$$

where μ_{x_1} represents the Haar measure on M and $\overline{p(x_1)}$ denotes the complex conjugation of $p(x_1)$. Thus, the $S(D)$-valued inner product can be represented as

$${}_D\langle f, g\rangle = \sum_{w\in D} \langle f, \pi_w g\rangle e_{D,\alpha}(w). \tag{30}$$

Manin's quantum theta function Θ_D [4, 5] was defined via algebra valued inner product up to a constant factor,

$${}_D\langle f_T, f_T\rangle \sim \Theta_D, \tag{31}$$

where f_T used in the construction was a simple Gaussian theta vector

$$f_T = \mathrm{e}^{\pi \mathrm{i} x_1^t T x_1}, \qquad x_1 \in M, \tag{32}$$

with T be an $n \times n$ complex valued matrix. Manin required that the quantum theta function Θ_D defined in this way should satisfy the following condition under translation:

$$^\forall g \in D, \qquad C_g e_{D,\alpha}(g) x_g^*(\Theta_D) = \Theta_D \tag{33}$$

where C_g is an appropriately given constant, and x_g^* is a 'quantum translation operator' defined as

$$x_g^*(e_{D,\alpha}(h)) = \mathcal{X}(g, h) e_{D,\alpha}(h) \tag{34}$$

with some commuting function $\mathcal{X}(g, h)$ for $g, h \in D$. Requirement (33) can be regarded as the quantum counterpart of the second property of the classical theta function, (2).

4. Quantum theta functions—extended to holomorphic connections with constants

In this section, we show that Manin's approach for quantum theta function also holds for the case of a theta vector obtained from more general holomorphic connections with constants.

As in the classical theta function case, we first introduce an n-dimensional complex variable $\underline{x} \in \mathbb{C}^n$ with complex structure T explained in the previous section as

$$\underline{x} \equiv T x_1 + x_2 \tag{35}$$

where $x = (x_1, x_2) \in M \times \widehat{M}$. On the basis of the defining concept for quantum theta function (31), Manin defined the quantum theta function Θ_D as

$${}_D\langle f_T, f_T\rangle = \frac{1}{\sqrt{2^n \det(\operatorname{Im} T)}} \Theta_D \tag{36}$$

with f_T given by (32).

We begin with the $S(D)$-valued inner product (31) with a more general theta vector $f_{T,c}$ which appeared in [7, 16].

$${}_D\langle f_{T,c}, f_{T,c}\rangle = \sum_{h\in D} \langle f_{T,c}, \pi_h f_{T,c}\rangle e_{D,\alpha}(h) \tag{37}$$

where

$$f_{T,c}(x_1) = \mathrm{e}^{\pi \mathrm{i} x_1^t T x_1 + 2\pi \mathrm{i} c^t x_1}, \qquad c \in \mathbb{C}^n, \quad x_1 \in M, \tag{38}$$

and T is the complex structure mentioned before. Here, we define π of $\mathcal{G}$ on $L^2(M)$ as follows:

$$(\pi_{(y_1,y_2)} f)(x_1) = \mathrm{e}^{2\pi \mathrm{i} x_1^t y_2 + \pi \mathrm{i} y_1^t y_2} f(x_1 + y_1), \qquad \text{for} \quad x, y \in \mathcal{G} = M \times \widehat{M}. \tag{39}$$

Then the cocycle $\alpha(x, y)$ in (26) is given by $\alpha(x, y) = \mathrm{e}^{\pi \mathrm{i}(x_1^t y_2 - y_1^t x_2)}$. From (29) and (39), the algebra valued inner product (37) can be written as

$$
\begin{aligned}
{}_D\langle f_{T,c}, f_{T,c}\rangle &= \sum_{h\in D} \langle f_{T,c}, \pi_h f_{T,c}\rangle e_{D,\alpha}(h) \\
&= \sum_{h\in D} \int_{\mathbb{R}^n} \mathrm{d}\mu_{x_1}\, f_{T,c}(x_1)\overline{(\pi_h f_{T,c})(x_1)} e_{D,\alpha}(h) \\
&\equiv \sum_{h\in D} \int_{\mathbb{R}^n} \mathrm{d}\mu_{x_1}\, \mathrm{e}^{-\pi[q(x_1)+l_{h,c}(x_1)+\widetilde{C}_{h,c}]} e_{D,\alpha}(h)
\end{aligned}
\tag{40}
$$

where $q(x_1)$, $l_{h,c}(x_1)$, $\widetilde{C}_{h,c}$ are defined by

$$
\begin{aligned}
q(x_1) &= 2x_1^t(\mathrm{Im}\, T)x_1, \\
l_{h,c}(x_1) &= 2\mathrm{i}x_1^t(\overline{T}h_1 + h_2 - 2\mathrm{i}(\mathrm{Im}\, c)), \\
\widetilde{C}_{h,c} &= \mathrm{i}h_1^t(\overline{T}h_1 + h_2 + 2\overline{c}).
\end{aligned}
\tag{41}
$$

Denoting

$$
\lambda_{h,c} \equiv \frac{\mathrm{i}}{2}(\mathrm{Im}\, T)^{-1}(\underline{h}^* - 2\mathrm{i}(\mathrm{Im}\, c)),
$$

one can check that

$$
q(x_1) + l_{h,c}(x_1) = q(x_1 + \lambda_{h,c}) - q(\lambda_{h,c}).
$$

Thus, the algebra valued inner product (40) can be written as

$$
{}_D\langle f_{T,c}, f_{T,c}\rangle = \sum_{h\in D} \mathrm{e}^{-\pi(\widetilde{C}_{h,c} - q(\lambda_{h,c}))} e_{D,\alpha}(h) \int_{\mathbb{R}^n} \mathrm{d}\mu_{x_1}\, \mathrm{e}^{-\pi q(x_1+\lambda_{h,c})}. \tag{42}
$$

Since $\int_{\mathbb{R}^n} \mathrm{d}\mu_{x_1}\, \mathrm{e}^{-\pi q(x_1+\lambda_{h,c})} = 1/\sqrt{\det q}$, the above expression can be rewritten as

$$
{}_D\langle f_{T,c}, f_{T,c}\rangle = \frac{1}{\sqrt{2^n \det(\mathrm{Im}\, T)}} \sum_{h\in D} \mathrm{e}^{-\pi(\widetilde{C}_{h,c} - q(\lambda_{h,c}))} e_{D,\alpha}(h) \tag{43}
$$

and we define our quantum theta function $\Theta_{D,c}$ as

$$
{}_D\langle f_{T,c}, f_{T,c}\rangle \equiv \frac{1}{\sqrt{2^n \det(\mathrm{Im}\, T)}} \Theta_{D,c}. \tag{44}
$$

The quantum theta function defined above is evaluated as

$$
\begin{aligned}
\Theta_{D,c} &= \sum_{h\in D} \mathrm{e}^{-\pi(\widetilde{C}_{h,c} - q(\lambda_{h,c}))} e_{D,\alpha}(h) \\
&= \sum_{h\in D} \mathrm{e}^{-\pi[\frac{1}{2}(\underline{h}^t - 2\mathrm{i}(\mathrm{Im}\, c)^t)(\mathrm{Im}\, T)^{-1}(\underline{h}^* - 2\mathrm{i}(\mathrm{Im}\, c)) + 2\mathrm{i}h_1^t\, (\mathrm{Re}\, c)]} e_{D,\alpha}(h).
\end{aligned}
\tag{45}
$$

Let $x^*_{g,c}$ be a 'quantum translation operator' defined by

$$
x^*_{g,c}(e_{D,\alpha}(h)) \equiv \mathrm{e}^{-\pi X(\underline{g},\underline{h})} e_{D,\alpha}(h), \tag{46}
$$

where $X(\underline{g}, \underline{h})$ is given by

$$
X(\underline{g}, \underline{h}) = \underline{g}^t(\mathrm{Im}\, T)^{-1}\underline{h}^* + 2(\mathrm{Im}\, c)^t(\mathrm{Im}\, T)^{-1}(\mathrm{Im}\, c).
$$

Then, for any element g in D the above quantum theta function $\Theta_{D,c}$ satisfies the following relation:

$$
C_{g,c} e_{D,\alpha}(g) x^*_{g,c}(\Theta_{D,c}) = \Theta_{D,c}, \tag{47}
$$

where

$$C_{g,c} \equiv \mathrm{e}^{-\frac{\pi}{2}H_c(\underline{g},\underline{g})}$$

and

$$H_c(\underline{g},\underline{g}) = (\underline{g} - 2\mathrm{i}(\mathrm{Im}\, c))^t(\mathrm{Im}\, T)^{-1}(\underline{g}^* - 2\mathrm{i}(\mathrm{Im}\, c)) + 4\mathrm{i}g_1^t(\mathrm{Re}\, c). \tag{48}$$

To prove relation (47) we first note that from (45) and (48) our quantum theta function $\Theta_{D,c}$ can be expressed as

$$\Theta_{D,c} = \sum_{h\in D} \mathrm{e}^{-\frac{\pi}{2}H_c(\underline{h},\underline{h})}\, \mathrm{e}_{D,\alpha}(h). \tag{49}$$

Thus the left-hand side of the functional relation (47) can be written as

$$\begin{aligned} C_{g,c}e_{D,\alpha}(g)x^*_{g,c}(\Theta_{D,c}) &= \mathrm{e}^{-\frac{\pi}{2}H_c(\underline{g},\underline{g})}e_{D,\alpha}(g)x^*_{g,c}\left(\sum_{h\in D}\mathrm{e}^{-\frac{\pi}{2}H_c(\underline{h},\underline{h})}e_{D,\alpha}(h)\right) \\ &= \sum_{h\in D}\mathrm{e}^{-\frac{\pi}{2}H_c(\underline{g},\underline{g})}\,\mathrm{e}^{-\frac{\pi}{2}H_c(\underline{h},\underline{h})}\,\mathrm{e}^{-\pi X(\underline{g},\underline{h})}e_{D,\alpha}(g)e_{D,\alpha}(h). \end{aligned}$$

Then using the cocycle relation (27)

$$e_{D,\alpha}(g)e_{D,\alpha}(h) = \alpha(g,h)e_{D,\alpha}(g+h) = \mathrm{e}^{\pi\mathrm{i}\,\mathrm{Im}(\underline{g}^t(\mathrm{Im}\, T)^{-1}\underline{h}^*)}e_{D,\alpha}(g+h),$$

we get

$$\mathrm{e}^{-\frac{\pi}{2}H_c(\underline{g},\underline{g})}\,\mathrm{e}^{-\frac{\pi}{2}H_c(\underline{h},\underline{h})}\,\mathrm{e}^{-\pi X(\underline{g},\underline{h})}\,\mathrm{e}^{\pi\mathrm{i}\,\mathrm{Im}(\underline{g}^t(\mathrm{Im}\, T)^{-1}\underline{h}^*)} = \mathrm{e}^{-\frac{\pi}{2}H_c(\underline{g}+\underline{h},\underline{g}+\underline{h})},$$

proving relation (47).

5. Conclusion

In this paper we explained how Manin's quantum theta functions emerge naturally from the state vectors on quantum (noncommutative) torus via the algebra valued inner product.

The theta vectors can be regarded as invariant state vectors under parallel transport on the noncommutative torus equipped with complex structures. However, they are not like the classical theta functions which are the state vectors (holomorphic sections of line bundles) over classical tori. The classical theta functions or kq representations are functions over the (complex n or real $2n$ dimensional) phase space consisting of coordinates and their canonical momenta, while the theta vectors are state functions over (real n dimensional) coordinates only.

In order to construct a quantum version of classical theta function, we need to build an operator function over the (real $2n$ dimensional) quantum phase space. We do this with a (commuting) state function via the algebra valued inner product. Thus the quantum theta function obtained via the algebra valued inner product from the theta vector (a function over commuting variables) can be regarded as a quantum version of the classical theta function or the kq representation.

In conclusion, we can say that the quantum theta function is a quantum version of the classical theta function which is equivalent to the kq representation, while the theta vector corresponds to a wavefunction over commuting coordinates, and the wavefunction in turn corresponds to the pre-transformed function for the kq representation.

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References

[1] Polchinski J 1998 *String Theory* vol 1 (Cambridge: Cambridge University Press)
[2] Green M B, Scwarz J H and Witten E 1987 *Superstring Theory* vol 2 (Cambridge: Cambridge University Press)
[3] Manin Y 1990 Quantized theta-functions *Common Trends in Mathematics and Quantum Field Theories (Kyoto, 1990) Prog. Theor. Phys. Suppl.* **102** 219
Manin Y 2000 Theta functions, quantum tori and Heisenberg groups *Preprint* math.AG/0011197
[4] Manin Y 2002 Real multiplication and noncommutative geometry *Preprint* math.AG/0202109
[5] Manin Y 2003 Functional equations for quantum theta functions *Preprint* math.QA/0307393
[6] Schwarz A 2001 *Lett. Math. Phys.* **58** 81
[7] Dieng M and Schwarz A 2002 Differential and complex geometry of two-dimensional noncommutative tori *Preprint* QA/0203160
[8] Martinec E J and Moore G 2001 Noncommutative solitons on orbifolds *Preprint* hep-th/0101199
[9] Gopakumar R, Minwalla S and Strominger A 2000 *J. High Energy Phys.* JHEP05(2000)020 (*Preprint* hep-th/0003160)
[10] Gopakumar R, Headrick M and Spradlin M 2003 *Commun. Math. Phys.* **233** 355 (*Preprint* hep-th/0103256)
[11] Boca F P 1999 *Commun. Math. Phys.* **202** 325
[12] Rieffel M 1988 *Can. J. Math.* **XL** 257
[13] Mumford D 1983/1984 *Tata Lectures on Theta I, II* (Basel-Boston: Birkhauser)
[14] Zak J 1972 *Solid State Physics* vol 27 ed H Ehrenreich, F Seitz and D Turnbull (New York: Academic)
[15] Bacry H, Grossmann A and Zak J 1975 *Phys. Rev.* B **12** 1118
[16] Kim H and Lee C-Y 2004 *J. Math. Phys.* **45** 461 (*Preprint* hep-th/0303091)